II. STABLE/UNSTABLE SYSTEM DECOMPOSITION ALGORITHM

The stable-unstable separation of a system using real Schur decomposition is a numerically stable procedure. The decomposition technique developed here contains two stages of transformations. In first stage the block form of Real Schur transformation [9] has been used. It involves determination of an unitary matrix \( U \) such that it transforms a given matrix \( A \) into \( A' = U^\top A U \), where \( A' \) is a block upper triangular Schur matrix and contains a coupling matrix. In second stage of transformation generalized Lyapunov equation (also called Sylvester Equation) is solved [8, pages 169-171] and completely decoupled stable and unstable subsystems are obtained.

The beauty of the algorithm is that the method presented is numerically an stable algorithm which separates in to two independent (decoupled) subsystems having all the eigen values same as that of original system. The brief steps of developed algorithm are as below:

1) Step1.

Let us consider an unstable system (continuous or discrete) as,

\[
G = \begin{bmatrix}
A & B \\
C & D
\end{bmatrix}
\]

(1)

Transform this system using an unitary matrix \( U \) in upper block diagonal Schur form [9], such that the eigen values of the transformed system is arranged in increasing order of its real parts (but in case of discrete system arranged in increasing order of absolute value of its eigen values).

If \( x \) is noted for original system states then the first stage transformation matrix \( U \) and the transformed system states \( x' \) may be related as,

\[
x = U x'
\]

Thus the first stage transformed matrix becomes,

\[
G = \begin{bmatrix}
U^\top A U & U^\top B \\
CU & D
\end{bmatrix} = \begin{bmatrix}
A' & B' \\
C' & D
\end{bmatrix}
\]
This system contains three stable eigenvalues and two unstable eigenvalues of the original system are contained in lower right block \( (A_{22}) \) and all the unstable eigenvalues of the original system are contained in lower left block \( (A_{11}) \).

2) \textit{step2:}

The transformed system in eqn(2), contains a coupling term \( A_{12} \). Therefore to bring eqn(2) into to completely decoupled form, we solve the general form of Lyapunov equation [8] using [9],

\[
A_{11}^{-1}S - S A_{22} + A_{12} = 0
\]

(3)

Solve eqn (3) and obtain the value of \( S \) and proceed for second stage of transformation,

\[ x_{t} = \tilde{W} \tilde{X} \]

where \( \tilde{X} \) is the final stage new transformed states and \( \tilde{W} \) is the final stage transformation matrix.

The second stage of transformation matrix \( \tilde{W} \) may be given as [8],

\[
\tilde{W} = \begin{bmatrix}
I_{m} & : & S \\
\vdots & \cdots & \cdots \\
O & : & I_{n}
\end{bmatrix}
\]

(4)

where, \( I_{m} & \text{ & } I_{n} = \text{Identity matrix of size m & n respectively.} \)

The important property of \( \tilde{W} \) is that its inverse may be obtained directly by simply replacing \( S \) with \( -S \).

Using \( \tilde{W} \), we get the completely decoupled system \( (\tilde{G}_d) \) as,

\[
\tilde{G}_d = \begin{bmatrix}
W^{-1}AW & W^{-1}B_1 \\
C & D
\end{bmatrix} = \begin{bmatrix}
a_{11} & O & a_{12} \\
O & a_{22} & b_1 \\
C & D & b_2
\end{bmatrix}_{m \times n}
\]

(5)

This transformed model may be decomposed in to stable and unstable subsystems as below:

\[
\tilde{G}_d = \begin{bmatrix}
a_{11} & b_1 \\
c_1 & D
\end{bmatrix} + \begin{bmatrix}
a_{22} & b_2 \\
c_2 & 0
\end{bmatrix}
\]

(6)

which is the final decomposed system in to two subsystems.

### III. APPLICATION OF UNSTABLE SYSTEM DECOMPOSITION TO REDUCED ORDER MODELING:

If any model reduction algorithm holds good only for stable system then also we can utilize that particular algorithm to reduce unstable system after system decomposition in which decomposed stable part is reduced and decomposed unstable part is directly added to this to get final reduced order model.

For example, in model reduction using balanced realization technique, the Lyapunov equations only exist for stable systems, therefore the algorithm applicable for the stable system is directly not possible for unstable system. But we can still utilize that algorithm to reduce unstable system provided the original system is decomposed in to two parts as below:

Original system\(=\)Completely Stable subsystem

\( + \) Completely Unstable subsystem

After decomposition, the completely stable subsystem is reduced (using any model reduction algorithm) and then adding to this the already separated completely unstable part just in the style of Glover [1, pages 1177-1178] gives the final reduced order model of the given unstable system.

Thus, the final reduced order model for unstable system is obtained by reducing the decomposed stable part of above equation and then directly adding it to the decomposed unstable part as below :

\[
G_R = G_s(\text{reduced}) + G_u(\text{unstable})
\]

(7)

### IV. ILLUSTRATIVE EXAMPLES:

A number of MIMO & SISO (continuous & discrete) LTI problems were studied for system decomposition & its order reduction. The response of reduced model obtained for each case is found in very close approximation to that of original system. Here, two problems are illustrated.

#### Example 1: DISCRETE SYSTEM

Here an unstable SISO discrete system of 4th order is taken from Al-Saggaff [7] as below:

\[
g(z) = \frac{2.2256(z - 2.0395)(z - 0.3460)(z + 0.3797)}{(z - 1.755)(z - 0.8605)(z + 0.5566)^2 + 0.1404}\]

This system contains three stable eigenvalues & two unstable eigenvalues.

For the purpose of decomposition, we convert the above system in to state space form as below :

\[
G = \begin{bmatrix}
A & B \\
C & D
\end{bmatrix} = \begin{bmatrix}
-5.566 & 1.0000 & -1.2472 & -0.8536 & 0 \\
-0.0197 & -0.5366 & 0.8283 & 0.5669 & 0 \\
0 & 0 & 1.7550 & 1.4611 & 0 \\
0 & 0 & 0 & 0.8605 & 2.9837 \\
3.0574 & 0 & 1.0898 & 0.7459 & 0
\end{bmatrix}
\]

From step 1, we get the first stage transformation matrix \( U \), and upper block diagonal Schur matrix \( A \), as below:
Here, apart from this example of minimal system it is obtained as below:

Finally, the stable and unstable part of the original system is obtained as below:

where, the eigen values of stable subsystem \( G_s \) are 
-0.5566 ± j0.1404 , \( 0.8605 \) and, the eigen value of unstable subsystem \( G_u \) is 1.7550

In pole zero form we can represent the above decomposed discrete system as,

In pole zero transfer function form, we have

The dc gain of this reduced model is \( =8.1141 \) also, the dc gain of original system \( =8.1141 \)
while, the dc gain of truncated model \( =8.1863 \)

Here, apart from this example of minimal system it is very interesting to note that we can also maintain the dc gain of reduced model of any non-minimal MIMO system (provided its dc gain exists) using first getting structured minimal realization (in which those states which are not connected with input and output both, are eliminated) of its stable subsystem and then its reduction and then preserving its dc gain and finally adding the already separated unstable subsystem .The examples have been tried but not being presented here because of space limitations . This may be thought a further investigation on extension of balanced realization based model reduction technique for any nonminimal (stable or unstable, continuous or discrete) system with preservation of its dc gain.

Now let us compare the reduced model obtained above using system decomposition with that of Al-Saggaff [10],

The step responses of models and also the step responses of modeling errors are plotted in fig.1 and in fig. 2 respectively.

It is noted that the dc gain of the balanced truncated model may not be equal to that of the original system. During the reduction of stable part we may preserve the dc gain of the stable part (may be called balanced residualized model), and if we directly add to this the decomposed unstable part, we find the resulting model will also preserve the dc gain of original system as below:
The model obtained by Al-Saggaff [7] possess much higher time domain error as compared to that of reduction using system decomposition which is observed from the plot of figure(2). It should be noted that in reduced model of the model reduction using system decomposition the unstable eigen value/values are remain unchanged while in the model obtained by Al-Saggaff [7] the unstable eigen value has been modified/changed during reduction.

Thus it indicates with the remarkable conclusion that we should always prefer the eigen value of unstable part to be kept as it is and not to modify/change the eigen value of unstable component while reduction.

The nature of step responses of proposed models using system decomposition based on balanced realization and their modeling errors shows the advantage and usefulness of the model over the model as obtained in referred paper[7] using generalized normal representation method.

The priori theoretical $L_{\infty}$ frequency domain error bound in both balanced realization based models using system decomposition is, $= 0.3289$

The priori theoretical $L_{\infty}$ frequency domain error bound as calculated in Al-Saggaff [7] using generalized normal representations is, $= 0.446$

Taking $H_{\infty}$ norms of different modeling errors transfer functions, we obtain the actual frequency domain error bound in modeling are shown in figure(3).

In the upper frequency range the upper bound of the modeling error in truncated model is lowest while in lower frequency range error bound is lowest for model preserving DC gain. The frequency domain (not time domain) error bound in [7] lie in between these two.

The truncated model possess minimum error in high frequency region, therefore such models are more appropriate for high frequency applications. Similarly the case is reverse for balanced resudalized model which will be more appropriate for low frequency applications. The applications of the model [7] in intermediate frequency range may be suitable but possessing much time domain error.

**Example 2: CONTINUOUS SYSTEM**

Consider the NASA HIMIT Fighter Design Data[9],

\[
A = \begin{bmatrix}
-0.22567 & -36.617 & -10.8897 & -32.090 & 3.2509 & -0.76257 \\
9.25726 & -1.8997 & 0.98312 & -7.2562e04 & -0.17080 & -4.965263 \\
0.012338 & 11.720 & -2.6316 & 8.758204 & -31.604 & 22.396 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -30 & 0 \\
0 & 0 & 0 & 0 & 0 & -30
\end{bmatrix}
\]

\[
B = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 30 \\
0 & 0 & 0 & 0 & 0 & 30 \\
0 & 0 & 0 & 0 & 0 & 30
\end{bmatrix}
\]

\[
C = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 30 \\
0 & 0 & 0 & 0 & 0 & -30
\end{bmatrix}
\]

\[
D = \begin{bmatrix}
0 & 0 \\
0 & 0
\end{bmatrix}
\]

This system contains four stable eigen values & two unstable eigen values. Following the proposed algorithm, we find the decomposed stable part ($G_s$) and unstable part ($G_u$) of the original system as below:

\[
G_s = \begin{bmatrix}
3.0009e01 & -1.6952e01 & 3.251e00 & 3.5623e00 & -1.0454e02 & 5.873e00 \\
-3.4899e14 & -3.0000e00 & -1.2769e01 & -3.2320e01 & 2.873e00 & -5.3615e00 \\
8.1368e014 & -1.9787e01 & -5.675e00 & -2.543e00 & 7.5169e00 & 4.988e00 \\
-1.5402e13 & 4.0500e11 & 1.7495e01 & -2.5779e01 & 1.918e00 & -2.193e00 \\
2.1334e00 & -1.1654e00 & 3.3337e01 & 1.6545e01 & 0 & 0 \\
2.3871e02 & -2.6018e01 & -1.343e00 & -1.325e00 & 0 & 0
\end{bmatrix}
\]

\[
G_u = \begin{bmatrix}
5.0388e-02 & -2.4547e+00 & -3.2300e+00 & 2.3076e+00 \\
1.9177e+01 & 1.3292e+00 & 7.986e+00 & -5.488e+00 \\
1.5547e-02 & -5.1119e-01 & 0 & 0 \\
4.5442e-01 & -4.4971e-01 & 0 & 0
\end{bmatrix}
\]

where, the eigen values of stable subsystem ($G_s$) are

-30.0000, -30.0000, -5.6757, -0.2578.

and, the eigen value of unstable subsystem ($G_u$) are

0.6898+j0.2488, 0.6898 - j0.2488.

We get the final 4th order balanced truncated reduced model and 4th order balanced resudalised model respectively as,
The step response of the original system and its reduced models, are shown in Fig. (4) & Fig. (5).

\[
Gr_{tr} = \begin{bmatrix}
-30.2400 & -3.4605 & 0 & 0 & 0 & 0 & -0.3144 & -29.9457 \\
0.6685 & -0.2179 & 0 & 0 & 0 & 0 & 1.5464 & -0.3520 \\
0 & 0 & 0.3520 & -2.3200 & 2.3076 \\
0 & 0 & 0.0504 & -2.4547 & 7.8964 & -5.4880 \\
0.1692 & 0.0719 & 0.0155 & -0.5112 & 0 & 0 \\
-29.9469 & -1.5843 & 0.4544 & -0.4497 & 0 & 0 \\
\end{bmatrix}
\]

\[
Gr_{dc} = \begin{bmatrix}
-30.9533 & -3.2853 & 0 & 0 & 0 & 0 & -1.0704 & -29.9457 \\
0.5834 & -0.1971 & 0 & 0 & 0 & 0 & 1.4563 & -0.3919 \\
0 & 0 & 0.0504 & -2.4547 & 7.8964 & -5.4880 \\
0 & 0 & 1.3292 & 7.8964 & 7.8964 & -5.4880 \\
1.0411 & 0.1330 & 0.0155 & -0.5112 & 0.8895 & 0.4086 \\
-30.2806 & -1.5022 & 0.4544 & -0.4497 & -0.3541 & -0.1562 \\
\end{bmatrix}
\]

**V. CONCLUSIONS:**

The stable/unstable system decomposition algorithm is developed for continuous and discrete system which does not require bilinear transformation. The algorithm is fully implementable using MATLAB package. The proposed algorithm is applied to reduce the unstable systems using balanced realization theory. The effectiveness of the algorithms is illustrated by examples.

The balanced truncated and balanced residualised model (model preserving dc gain) is obtained and frequency domain priori $L_\infty$ error bound and posteriori actual $L_\infty$ error bound in model order reduction has been calculated. The response of reduced order model is matching well with that of original system.

**VI. REFERENCES:**


